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DECOMPOSING HOPF ALGEBRAS OVER A FIELD OF NON-ZERO CHARACTERISTIC

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Let $\mathbf{A}$ be a connected, bi-associative, bi-commutative, graded Hopf algebra of finite type over a perfect field $\mathbf{K}$ of non-zero characteristic p. We will establish a decomposition theorem for $\mathbf{A}$ which can be useful in computing the coalgebra structure in the Pontrjagin rings of certain loop spaces and which does not appear to be present in the literature. A related theorem is given in [1].

Theorem 1. (a) $(p=2)$ *For each odd integer* s *there exists a sub-Hopf algebra* $\mathbf{C}(s)$ *of* $\mathbf{A}$ *generated as an algebra by classes with dimensions* $2^\alpha s$ *with* $\alpha \geq 0$ *such that the tensor product of the inclusions gives an isomorphism of Hopf algebras* $\otimes\{\mathbf{C}(s)\} \cong \mathbf{A}$.

(b) (p odd) *For each integer* s *prime to* p *there exists a sub-Hopf algebra* $\mathbf{C}(2s)$ *of* $\mathbf{A}$ *generated as an algebra by classes with dimensions* $2p^\alpha s$ *with* $\alpha \geq 0$ *and for each odd integer* $2r+1$ *a sub-Hopf algebra* $\mathbf{B}(2r+1)$ *which is an exterior algebra on classes with dimensions* $2r+1$ *such that the tensor product of the inclusions gives a Hopf algebra isomorphism* $\{\otimes\mathbf{B}(2r+1)\}\otimes\{\otimes\mathbf{C}(2s)\} \cong \mathbf{A}$.

(c) *The factors* $\mathbf{C}(s)$ *of* (a) *and* $\mathbf{B}(2r+1)$ *and* $\mathbf{C}(2s)$ *of* (b) *are defined uniquely by the properties described.*

The proofs of (a) and (b) are similar and so we establish the latter only.

First we recall from Proposition 4.23 of [2] that there is an exact sequence

$$\mathbf{O} \longrightarrow \mathbf{P}(\mathbf{K}(\xi(\mathbf{A}))) \longrightarrow \mathbf{P}(\mathbf{A}) \longrightarrow \mathbf{Q}(\mathbf{A}) \longrightarrow \mathbf{Q}(\mathbf{K}(\lambda(\mathbf{A}))) \longrightarrow \mathbf{O}. \tag{1}$$

Suppose that k is a positive integer such that sub-Hopf algebras $\mathbf{B}(2r+1)$ and $\mathbf{C}(2s)$ of $\mathbf{A}$ can be found with generators in the appropriate dimensions and all s prime to p such that the tensor product of the inclusions induce an isomorphism of the Hopf algebras $\{\otimes\mathbf{B}(2r+1)\}\otimes\{\otimes\mathbf{C}(2s)\}$ and $\mathbf{A}$ in dimensions less than k. We shall prove that each $\mathbf{B}(2r+1)$ and $\mathbf{C}(2s)$ is well defined by $\mathbf{A}$ in dimensions less than k.

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If $2r+1<k$, then $\mathbf{Q}(\mathbf{B}(2r+1))=\mathbf{Q}(\mathbf{A})_{2r+1}$ and from (1) $\mathbf{P}(\mathbf{A}_{2r+1})\cong\mathbf{Q}(\mathbf{A})_{2r+1}$ for dimensional reasons. As $\mathbf{B}(2r+1)$ is a sub-Hopf algebra, we must have $\mathbf{B}(2r+1)_{2r+1}=\mathbf{P}(\mathbf{A}_{2r+1})$. Now it is routine to verify or to deduce from Proposition 7.10 of [2] that $\mathbf{B}(2r+1)$ is the exterior algebra generated by $\mathbf{P}(\mathbf{A}_{2r+1})$.

If $2s<k$, then again we know that $\mathbf{Q}(\mathbf{C}(2s))_{2s}=\mathbf{Q}(\mathbf{A})_{2s}$ and from (1) $\mathbf{P}(\mathbf{A}_{2s})\cong\mathbf{Q}(\mathbf{A})_{2s}$ and so $\mathbf{C}(2s)_{2s}=\mathbf{P}(\mathbf{A}_{2s})$. We assume inductively that we have shown that $\mathbf{C}(2s)$ is uniquely defined in dimensions less that $t<k$. Suppose that there is another decomposition of $\mathbf{A}$ in dimensions less than k giving rise to $\mathbf{C}'(2s)$ where $\mathbf{C}(2s)$ and $\mathbf{C}'(2s)$ differ in dimension t. There must exist an indecomposable $u\in\mathbf{C}(2s)_t$ and a non-zero decomposable $v\in\mathbf{I}$ such that $u+v\in\mathbf{C}'(2s)_t$, where $\mathbf{I}$ is the Hopf ideal generated by all $\overline{\mathbf{B}(2r+1)}$ and $\overline{\mathbf{C}(2s')}$ in the first decomposition with $s\neq s'$. Our induction hypothesis implies that $\psi(u+v)-(u+v)\otimes 1-1\otimes(u-v)$ lies in $\overline{\mathbf{C}(2s)}\otimes\overline{\mathbf{C}(2s)}$ and as $\psi(u)-u\otimes 1-1\otimes u\in\overline{\mathbf{C}(2s)}\otimes\overline{\mathbf{C}(2s)}$, it follows that $\psi(v)-v\otimes 1-1\otimes v$ lies in $\overline{\mathbf{C}(2s)}\otimes\overline{\mathbf{C}(2s)}$. We now need a lemma.

Let $\mathbf{K}$ be a perfect field so that $\mathbf{K}\xi(\mathbf{A})=\xi(\mathbf{A})$. The following result is a well known consequence of (1).

Lemma 2. *Let $v\in\mathbf{P}(\mathbf{A}_k)$ be decomposable in $\mathbf{A}$. Then $v=w^{p^r}$ for some $r>0$ where w is indecomposable in $\mathbf{A}$.*

Returning to the v above, dimensional reasons imply that $v\neq w^{p^r}$ for any indecomposable w and so by the lemma v is not primitive. Hence $\psi(v)-v\otimes 1-1\otimes v$ is non-zero in $(\bar{\mathbf{A}}\otimes\mathbf{I}+\mathbf{I}\otimes\bar{\mathbf{A}})_t$. But

$$(\overline{\mathbf{C}(2s)}\otimes\overline{\mathbf{C}(2s)})_t\cap(\bar{\mathbf{A}}\otimes\mathbf{I}+\mathbf{I}\otimes\bar{\mathbf{A}})_t=\mathbf{O}$$

and we have a contradiction. Thus $\mathbf{C}(2s)$ is defined uniquely in dimensions less than k.

We now choose the integer k to be as large as possible such that for all $\mathbf{A}$ a decompsotion exists as above in dimensions less than k. As $\mathbf{A}$ is connected, $k>1$. If no finite k exists, then (b) is proved. If k is finite, we choose an example of $\mathbf{A}$ for which a decomposition of the required type does not exist in dimensions less than $k+1$ and in which $\dim\mathbf{A}_k$ is minimum. We will obtain a contradiction.

Let $\mathbf{D}=\{\otimes\mathbf{B}(2r+1)\}\otimes\{\otimes\mathbf{C}(2s)\}$ be the unique decomposition of $\mathbf{A}$ in dimensions less than k where we assume without loss of generality that none of the Hopf algebras $\mathbf{B}(2r+1)$ and $\mathbf{C}(2s)$ have indecomposables with dimensions greater than $k-1$. We show first that the inclusions induce a monomorpism $i:\mathbf{D}\to\mathbf{A}$ in dimensions less than $k+1$. Suppose that $i(u)=0$ in $\mathbf{A}$ where u has dimension k. Then u is primitive and decomposable in $\mathbf{D}$. Hence by Lemma 2, $u=w^{p^r}$ where w is indecomposable. If $\dim w=2p^\alpha s$, we write $w=t+z$ where $t\in\mathbf{C}(2s)$ and $z\in\mathbf{I}$ in the notation used above. Arguing as in the proof of the uniqueness of $\mathbf{C}(2s)$ we deduce that $z^{p^r}=0$. Thus $u=t^{p^r}\in\mathbf{C}(2s)$. But the restriction of i to $\mathbf{C}(2s)$ is a monomorphism. Hence $i(u)=0$ implies that $u=0$.

We now show that i can be extended to an isomorphism. This is clearly the case unless $\mathbf{Q(A)}_k \neq \mathbf{O}$ and there exists $\bar{u} \in \mathbf{Q(A)}_k$ which has no representative $u \in \mathbf{A}_k$ with $\psi(u) - u \otimes 1 - 1 \otimes u$ lying in $\mathbf{C}(2s) \otimes \mathbf{C}(2s)$ with s determined by k. Also $\mathbf{P}(\mathbf{A}_k) = \mathbf{O}$ for if $v \in \mathbf{P}(\mathbf{A}_k)$ is non-zero and $\mathbf{J}$ is the Hopf ideal which it generates, $\mathbf{A/J}$ could be considered in place of $\mathbf{A}$ and $\dim(\mathbf{A/J})_k = \dim \mathbf{A}_k - 1$.

But $\mathbf{A}^*$, the dual Hopf algebra of $\mathbf{A}$, must also satisfy $\mathbf{Q}(\mathbf{A}^*)_k \neq \mathbf{O}$. Otherwise $\mathbf{A}^*$ decomposes as $\{\otimes \mathbf{B}(2r+1)\} \otimes \{\otimes \mathbf{C}(2s)\}$ in dimensions less than $k+1$. Then $\mathbf{A}$ is abstractly isomorphic to $\{\otimes \mathbf{B}^*(2r+1)\} \otimes \{\otimes \mathbf{C}^*(2s)\}$ in dimensions less than $k+1$. Clearly $\mathbf{B}^*(2r+1)$ is an exterior algebra on generators with dimensions $2r+1$. Also $\mathbf{C}^*(2s)$ is generated by classes with dimensions $2p^\beta s$ since $\mathbf{Q}(\mathbf{C}^*(2s)) \cong \mathbf{P}(\mathbf{C}(2s))^*$ and by Lemma 2 primitives of $\mathbf{C}(2s)$ lie in dimensions $2p^\beta s$ as all indecomposables lie in dimensions $2p^\alpha s$. From this a decomposition of $\mathbf{A}$ of the required type can be constructed in dimensions less than $k+1$ which is not possible by hypothesis. We have therefore $\mathbf{O} = \mathbf{P}(\mathbf{A}_k)^* \cong \mathbf{Q}(\mathbf{A}^*)_k \neq \mathbf{O}$. This contradiction shows that k is not finite.

The uniqueness of the decomposition given in (c) follows by induction as in the argument used to establish uniqueness in dimensions less than k.

Finally we remark that the hypothesis that K is perfect can be eliminated with a little extra work.

References

[1] J.R. Hubbuck, A Hopf algebra decomposition theorem, Bull. Lond. Math. Soc. 13 (1981) 125–128.
[2] J. Milnor and J. Moore, On the structure of Hopf algebras, Annals of Math. 81 (1965) 211–264.